Algebraic approach to parafermionic conformal field theories

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Abstract

Parafermionic conformal field theories are considered on a purely algebraic basis. The generalized Jacobi type identity is presented. Systems of free fermions coupled to each other by nontrivial parafermionic type relations are studied in detail. A new parafermionic conformal algebra is introduced, it describes the $sl(2|1)_2/u(1)^2$ coset system.

Contents

1 Introduction

The first parafermionic two-dimensional conformal field theory was introduced in 1985 in the classical article by V. Fateev and A. Zamolodchikov [1]. Parafermion fields have fractional conformal dimension and are not required to be local to each other, the order of their mutual singularity can be any real number. Parafermionic algebras can be seen as a generalization of standard conformal chiral algebras (vertex algebras in mathematical literature) to the case of mutually nonlocal fields.

In that paper [1] the authors study the $\mathbb{Z}_N$ invariant parafermionic conformal field theory with parafermion conformal dimensions being $\Delta_i = i(N-i)/N$ and show that this theory is equivalent to the $sl(2)_N/u(1)$ coset. Shortly after that in [2] the same authors used the $\mathbb{Z}_N$ parafermions to build the minimal models of the $N=2$ superconformal algebra and in [3] they studied a new $\mathbb{Z}_3$ parafermionic theory generated by currents of dimension 4/3.

In 1987 D. Gepner [4] introduced new parafermionic algebras describing coset theories of type $\mathfrak{g}/u(1)^r$, where $\mathfrak{g}$ is any simple Lie algebra and r is its rank. Conformal dimensions, fusion rules modular properties and partition functions were found in this paper. However the exact algebraic structure of the theory (e.g. structure constants) remained unclear.

Later on parafermions (mainly the classical $\mathbb{Z}_N$ parafermions from [1]) were widely used in different areas of conformal field theory and string theory. A search for new $\mathbb{Z}_N$ invariant parafermionic conformal field theories was a subject of a number of papers during the last 15 years, we would like to mention some of them. Furlan et.al. [5] studied

general $\mathbb{Z}_N$ invariant parafermionic theories using the correlation function technique. "Graded parafermions" were introduced in [6], these are based on the $osp(1|2)/u(1)$ coset construction. Parafermions of the $A_2^{(2)}/u(1)$ coset were studied in [7], their algebra is also graded by the $\mathbb{Z}_N$ group. Then in the recent years in series of papers [8, 9, 10, 11, 12] Dotsenko et.al. studied the second and the third solutions for the $\mathbb{Z}_N$ parafermion algebra from the Appendix to the paper [1]. And finely last year Jacob and Mathieu [13, 14] studied a new possibility for the $\mathbb{Z}_N$ algebra (N even): the parafermion dimensions being $\Delta_i = 3i(N-i)/2N$.

A unified algebraic description of parafermionic conformal field theories was still missing in the physical literature. The aim of this paper is to develop purely algebraic approach to conformal algebras of parafermionic type, and to illustrate its power on a few simple interesting examples. The presented Jacobi-type condition involves only operator product expansion relations. This generalized Jacobi identity in our opinion is the simplest tool to check the selfconsistence of conformal algebras. It allows to use the so called bootstrap approach in construction of new parafermionic theories, the same approach which was used to build new W-algebras in the early years of their study.

After the first version of this paper appeared on the web we learned about the theory of generalized vertex algebras developed in the mathematical literature. The most important reference is a monograph by C. Dong and J. Lepowsky [15] [1] in which conformal algebras of parafermionic type (generalized vertex algebras) are introduced for the first time from the mathematical point of view. The first part of our paper (Sections 2 and 3) has substantial overlap with the theory developed in the book. In particular the crucial notion of commutation factor and the generalized Jacobi identity are present in [15]. There are a few levels of generalization in the monograph, we suppose that our description of parafermionic conformal algebras corresponds to the most general concept in the book - abelian intertwining algebra (chapters 11, 12 of [15]).

Mathematically oriented readers would also be interested to consult a recent work by B. Bakalov and V. Kac [16]. In this paper generalized vertex algebras are introduced using the notion of polylocal fields. The definition of a generalized vertex algebra here is slightly more general than the one from [15]. The "Borcherds identity" (formula (27) in [16]) has exactly the same form as our generalized Jacobi identity (20) up to a difference in the choice of normalization of the commutation factors.

Contrary to the references [15] and [16] we focus here on the applications of the algebraic approach to the study of parafermionic conformal field theories. We employ this approach for the analysis of a few relatively simple but instructive examples of parafermionic algebras, for which we present the full set of algebraic relations, calculate the structure constants and the commutation factors, discuss the representation theory.

The paper has the following structure. In Section 2 we fix notations, define conformal algebras of parafermionic type and introduce the notion of commutation factor which is crucial for further developments and discuss consistency relations for commutation

[1] We are grateful to J. Lepowsky for letting us know about the monograph.

factors. In Section 3 the Jacobi-type condition is derived. The general commutator formula is stated in Section 4. Next four sections are devoted to examples of parafermionic algebras. The first warmup example is the classical $\mathbb{Z}_N$ invariant parafermion algebra. Next we study the algebra of $sl(3)$ fermions, formed by three dimension 1/2 fields (associated with the roots of the $sl(3)$ Lie algebra) coupled to each other in nontrivial way. The third example is the generalization of the second one to the $sl(n)$ case. And the last example is the algebra of the $sl(2|1)_2/u(1)^2$ coset, this algebra is generated by four dimension 1 fields and one free fermion field. A short summary is given in Section 9.

2 Framework and notation

In two-dimensional conformal field theory the conformal algebra is generated by a set of conformal fields $\phi_i(z)$ of given conformal dimensions $\Delta(\phi_i)$. The algebraic relations between the fields are operator product expansions. We refer the reader to the standard texts on conformal field theory and vertex algebras [17, 18, 19, 20, 21].

We will deal here with parafermionic conformal field theories, in which the operator product expansion of any two fields has the following form:

$$A(z)B(w) = \frac{1}{(z-w)^\alpha}\Big(\big[AB\big]_\alpha(w) + \big[AB\big]_{\alpha-1}(w)(z-w) + \big[AB\big]_{\alpha-2}(w)(z-w)^2 + \cdots\Big), \tag{1}$$

i.e. it is a general operator product expansion with one important restriction that except the overall singularity $(z-w)^{-\alpha}$ the integer powers of $(z-w)$ only are present in the brackets on the right hand side of the equation. The singularity $\alpha = \alpha(A, B)$ depends on the fields A and B, the dependence is always assumed even when not written explicitly. This singularity can be calculated from conformal dimensions as

$$\alpha = \Delta(A) + \Delta(B) - \Delta(\big[AB\big]_\alpha). \tag{2}$$

The expression $\big[AB\big]_n$ is the so called n-product of fields A and B. It is the field, arising at the $(z-w)^{-n}$ term of the operator product expansion of the fields $A(z)$ and $B(w)$ around w as it appears in (1). We usually assume that the most singular term $\big[AB\big]_\alpha$ is not zero.

The singularity α doesn't have to be integer. It can be any real number but normally it is rational. The standard conformal algebras (e.g. Virasoro, affine Kac-Moody, W-algebras) are also of parafermionic type according to our definition, but all the singularities are integer. So we study here a more general theory, and we will focus on the case when the operator product expansion singularities are not integer.

We should comment that parafermionic conformal field theories are not the most general conformal field theories. There are theories in which the powers of $(z-w)$ in the given operator product expansion differ by a non-integer number. The exact algebraic meaning of such relations is yet to be understood.

In order to see that the algebra is of parafermionic type one should choose an appropriate basis of fields, since an operator product expansion which looks like (1) in one basis can be of mixed form in another basis.

Now we want to introduce the mode expansions of the fields:

$$A(z) = \sum_{n \in -\Delta(A)+\epsilon(A)+\mathbb{Z}} A_n z^{-n-\Delta(A)}. \tag{3}$$

So we keep the standard notation for the modes, which is very convenient from the physical point of view: in this convention positive modes are annihilation operators and negative modes are creation operators. The sector of the representation is encoded in the *twisting* $\epsilon \in \mathbb{R}/\mathbb{Z}$. If $\epsilon(A) = 0$ then the powers of z are integer in (3) and we say that the field $A(z)$ is in the untwisted sector. Not any value of the twisting is allowed, it should be consistent with the operator product expansion relations. We should note that in the case of parafermionic algebras the action of a field mode on a state changes the sector of the state.

The formula (3) is reverted as following:

$$A_n = \oint_0 \mathrm{d}z\ A(z) z^{n+\Delta(A)-1}, \tag{4}$$

where $\oint_0$ means integration around zero, $\frac{1}{2\pi \mathrm{i}}$ is always assumed although is not written explicitly.

One should be able to express the reversed operator product expansion $B(w)A(z)$ in terms of the operator product expansion $A(z)B(w)$ itself. In the case of standard conformal algebras ($\alpha \in \mathbb{Z}$) the rule is $B(w)A(z) = -A(z)B(w)$ if both A and B are fermionic and $B(w)A(z) = A(z)B(w)$ if at least one of them is bosonic. When $\alpha \notin \mathbb{Z}$ it is not clear a priori how to exchange the fields in the operator product expansion, since some phases are involved. We overcome this difficulty by multiplying the operator product expansion by its most singular term. So we introduce the following axiom:

$$A(z)B(w)(z-w)^{\alpha} = \mu_{AB} B(w)A(z)(w-z)^{\alpha}. \tag{5}$$

This equation is also a definition[2] of the *commutation factor* μ_{AB} which is a complex number different from zero, and normally its absolute value is 1.

We add the axiom (5) to our definition of the parafermionic conformal algebra. This axiom is a key point of the theory, once one accepts it, all the following derivations are obtained almost automatically.

Now we are ready to clarify the exact meaning of an operator product expansion (1). There are two approaches: mathematical and physical. In the mathematical approach a field is a formal power series of type (3), where z is just a formal variable, then the

[2] The definition of parafermionic mutual locality given in (8) of [16] is essentially the same, the only difference is in the normalization of the commutation factor.

operator product expansion of two fields is just a product of two power series, but when one rearranges some modes in the product, power terms in $(z-w)$ appear (see e.g. [20] for details). In the physical approach the fields are physical quantum fields in a two-dimensional quantum field theory, and the variable is a complex variable, the real and imaginary parts of which are physical coordinates. Then in the operator product expansion one always assumes the radial ordering (compare [18], equation (2.9)):

$$(z-w)^{\alpha} R(A(z)B(w)) = \begin{cases} (z-w)^{\alpha} A(z)B(w), & |z| > |w|, \\ \mu_{AB}(w-z)^{\alpha} B(w)A(z), & |z| < |w|. \end{cases} \tag{6}$$

One can use either of the approaches but sometimes one is more convenient than another.

In the case of standard ($\alpha \in \mathbb{Z}$) conformal algebras $\mu_{AB} = -(-1)^{\alpha}$ if both A and B are fermionic and $\mu_{AB} = (-1)^{\alpha}$ if at least one of them is bosonic. But the commutation factor is different from ± 1 in general, and we will see such examples in the next sections.

By exchanging the fields in (5) second time one shows that the commutation factor should satisfy the following consistency conditions:

$$\mu_{AB}\mu_{BA} = 1 \tag{7}$$

and if we assume that the term $\big[AA\big]_{\alpha_{AA}} \neq 0$ then it follows that

$$\mu_{AA} = 1. \tag{8}$$

However in some situations one should keep the possibility that $\mu_{AA} = -1$, but if this is the case then $\big[AA\big]_{\alpha_{AA}} = 0$, and then the real singularity exponent is actually $\alpha_{AA} - 1$ and not α_{AA}.

If the operator product expansion of two basic fields $B(w)$ and $C(v)$ gives a third one $D(v)$:

$$B(w)C(v) = \frac{D(v)}{(w-v)^{\alpha_{BC}}} + \cdots, \tag{9}$$

then exchanging another basic field $A(z)$ with $B(w)$ and then with $C(v)$ is essentially the same as exchanging $A(z)$ with $D(v)$. Therefore μ_{AD} is proportional to $\mu_{AB}\mu_{AC}$. The exact statement is

$$\mu_{AB}\,\mu_{AC} = \mu_{AD}\,r_{ABC}, \qquad r_{ABC} = (-1)^{\alpha_{AB}+\alpha_{AC}-\alpha_{AD}} = \pm 1. \tag{10}$$

It is also implicitly stated here that $\alpha_{AB} + \alpha_{AC} - \alpha_{AD} \in \mathbb{Z}$.

To prove the statement consider the following regular function in the 3 variables z, w, v:

$$R(z,w,v) = A(z)B(w)C(v)(z-w)^{\alpha_{AB}}(z-v)^{\alpha_{AC}}(w-v)^{\alpha_{BC}}. \tag{11}$$

Exchange the field $A(z)$ with $B(w)$ and then with $C(v)$ and then use the expansion (9) and take a limit $w \to v$, the result is

$$\lim_{w\to v} R(z,w,v) = \mu_{AB}\mu_{AC} D(v)A(z)(v-z)^{\alpha_{AB}+\alpha_{AC}}. \tag{12}$$

Since this function should also be regular in v around z we conclude that $\alpha_{AB} + \alpha_{AC} - \alpha_{AD} \in \mathbb{Z}$. Now let us first use the expansion (9) and then exchange the fields $A(z)$ and $D(v)$ and then again take the limit $w \to v$ to get:

$$\lim_{w \to v} R(z, w, v) = \mu_{AD} D(v) A(z) (z - v)^{\alpha_{AB} + \alpha_{AC} - \alpha_{AD}} (v - z)^{\alpha_{AD}}. \tag{13}$$

Comparing the two expressions above we obtain the statement (10).

The condition (10) also follows from the generalized Jacobi identity stated in the next section and should not be checked once all the Jacobi identities are satisfied.

We should remark here that if we do not specify any field in the operator product expansion of $B(w)$ and $C(v)$, i.e.

$$B(w)C(v) = O((w - v)^{-\alpha_{BC}}), \tag{14}$$

then there is no restriction on the product $\mu_{AB}\mu_{AC}$ for any field A.

Conformal algebras of parafermionic type are connected in some sense to the so called color (or generalized, or "ϵ-") Lie algebras ([22, 23, 24, 25] and some later papers). These are Lie algebras with the bracket relation $[x, y] = xy - \epsilon_{xy} yx$, so the bracket is not symmetric or antisymmetric in general: $[x, y] = -\epsilon_{xy}[y, x]$.

3 Generalized Jacobi identity

Here we derive the analogue of the Jacobi identities for the conformal algebras of parafermionic type. But first we would like to express the $\big[BA\big]_n$ products in terms of $\big[AB\big]_n$ products. According to our axiom (5) the order of $A(z)$ and $B(w)$ in the operator product expansion is reversed as following

$$B(w)A(z) = \frac{\mu_{BA}}{(w - z)^{\alpha}} \Big(\big[AB\big]_{\alpha}(w) + \big[AB\big]_{\alpha-1}(w)(z - w) + \big[AB\big]_{\alpha-2}(w)(z - w)^2 + \cdots \Big), \tag{15}$$

and then one expands the fields in the second variable (we also switch $w \leftrightarrow z$ to get the convenient form):

$$\begin{aligned} B(z)A(w) = \frac{\mu_{BA}}{(z - w)^{\alpha}} \Bigg(& \big[AB\big]_{\alpha}(w) + \Big(\partial \big[AB\big]_{\alpha}(w) - \big[AB\big]_{\alpha-1}(w) \Big)(z - w) \\ & + \Big(\frac{1}{2!} \partial^2 \big[AB\big]_{\alpha}(w) - \partial \big[AB\big]_{\alpha-1}(w) + \big[AB\big]_{\alpha-2}(w) \Big)(z - w)^2 + \cdots \Bigg), \end{aligned} \tag{16}$$

so the $\big[BA\big]_x$ products are obtained from the $\big[AB\big]_x$ products as

$$\big[BA\big]_{\alpha-n} = \mu_{BA} \sum_{j=0}^{n} \frac{(-1)^j}{(n - j)!} \partial^{n-j} \big[AB\big]_{\alpha-j}. \tag{17}$$

It immediately follows that (if $\big[AA\big]_\alpha \neq 0$)

$$\big[AA\big]_{\alpha-1} = \frac{1}{2}\partial\big[AA\big]_\alpha, \tag{18}$$

and more generally

$$\big[AA\big]_{\alpha-n} = \frac{1}{2}\sum_{j=0}^{n-1}\frac{(-1)^j}{(n-j)!}\partial^{n-j}\big[AA\big]_{\alpha-j}, \qquad n\in 2\mathbb{Z}+1, \tag{19}$$

i.e. the pole $\big[AA\big]_{\alpha-n}$ for odd n is just a linear combination of derivatives of higher poles.

Now we proceed to the Jacobi type condition which involves three fields A, B and C. It is an analogue of the Jacobi identity for Lie algebras. The condition is:

$$\begin{aligned} &\sum_{j\geq 0}(-1)^j\binom{\gamma_{AB}}{j}\big[A\big[BC\big]_{\gamma_{BC}+1+j}\big]_{\gamma_{AB}+\gamma_{AC}+1-j} \\ -\mu_{AB}(-1)^{\alpha_{AB}-\gamma_{AB}}&\sum_{j\geq 0}(-1)^j\binom{\gamma_{AB}}{j}\big[B\big[AC\big]_{\gamma_{AC}+1+j}\big]_{\gamma_{AB}+\gamma_{BC}+1-j} \\ =&\sum_{j\geq 0}\binom{\gamma_{AC}}{j}\big[\big[AB\big]_{\gamma_{AB}+1+j}C\big]_{\gamma_{BC}+\gamma_{AC}+1-j}. \end{aligned} \tag{20}$$

All three sums are finite, the upper bound is given by the order of singularity of the corresponding fields. The parameters γ differ from the corresponding singularity exponents α by an integer number: $\alpha_{AB}-\gamma_{AB}, \alpha_{AC}-\gamma_{AC}, \alpha_{BC}-\gamma_{BC}\in\mathbb{Z}$. This identity is a modified version of the well known Borcherds identity, however an essential difference is the presence of the commutation factor μ_{AB}.[3]

We say that the theory is associative up to a certain order in the expansion of two fields A and B if the condition (20) is satisfied for the inner n-products being of this and more singular order. That means that if for any fields A, B one checks the identity (20) for $\gamma_{AB}\geq n_{AB}-1$ then the theory is self consistent up to the term $\big[AB\big]_{n_{AB}}(w)(z-w)^{-n_{AB}}$ in the operator product expansion of A and B.

The generalized Jacobi identity (20) is derived using the standard trick: double integrating the function

$$f(z,w) = A(z)B(w)C(0)(z-w)^{\gamma_{AB}}w^{\gamma_{BC}}z^{\gamma_{AC}} \tag{21}$$

[3] In mathematical literature a similar Jacobi type identity is introduced in [15]: the relation (11.9) in Chapter 11 “Intertwining operators” of the book contains the same information, although written in a different form. The Borcherds identity (27) in ref. [16] coincides with (20) above if one uses the same normalization of the commutation factor.

on z and w in two different ways: the first is the integration on w around 0, then on z around 0 minus the opposite order; the second is the integration on z around w, then on w around 0:

$$\oint_0 \mathrm{d}z \oint_0 \mathrm{d}w f(z,w) - \oint_0 \mathrm{d}w \oint_0 \mathrm{d}z f(z,w) = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z f(z,w), \tag{22}$$

there the internal (right) integral is taken first. Then the first integration gives the left hand side of (20), the second integration gives the right hand side.

In the case of usual conformal algebras (all the mutual singularities α are integer) take $\gamma_{AB} = 0$, $\gamma_{BC} = p-1$, $\gamma_{AC} = q-1$, where $p, q \in \mathbb{Z}$ to get[4] the Jacobi identity:

$$\big[A\big[BC\big]_p\big]_q - (-1)^{|A||B|}\big[B\big[AC\big]_q\big]_p = \sum_{j=0}^{\infty} \binom{q-1}{j} \big[\big[AB\big]_{1+j} C\big]_{p+q-1-j}, \tag{23}$$

which can be found elsewhere (see for example [26], formula (2.3.21)). $|A|$ and $|B|$ are parities of the fields. This formula is also valid when only α_{AB} is integer, and two other singularities are arbitrary. (p and q are not integer any more in this case.)

4 Commutator formula

Here we derive the generalized commutator formula for the modes of two fields, the operator product expansion of which is of parafermionic type.

In this section it will be convenient to change the notation. We will consider here only one operator product expansion relation:

$$A(z)B(w) = \frac{1}{(z-w)^\alpha}\Big(C^{(0)}(w) + C^{(1)}(w)(z-w) + C^{(2)}(w)(z-w)^2 + \cdots\Big). \tag{24}$$

So comparing to (1) we have changed: $C^{(j)} = \big[AB\big]_{\alpha-j}$.

The main statement of this section is that the operator product expansion (24) is equivalent to the following commutator formula for the field modes:

$$\sum_{j=0}^{\infty} \binom{\alpha-k}{j} (-1)^j A_m B_{n-m} \Big|_{m=-\Delta_A+\alpha-k+\gamma+1-j} - \mu_{AB}(-1)^k \sum_{j=0}^{\infty} \binom{\alpha-k}{j} (-1)^j B_{n-m'} A_{m'} \Big|_{m'=-\Delta_A+\gamma+1+j} = \sum_{l=0}^{k-1} \binom{\gamma}{k-1-l} C_n^{(l)}. \tag{25}$$

Changing $\gamma \to m + \Delta_A - 1, n \to n+m$ we get a different form which is usually more convenient:

$$\sum_{j=0}^{\infty} \binom{\alpha-k}{j} (-1)^j \Big(A_{m+\alpha-k-j} B_{n-\alpha+k+j} - \mu_{AB}(-1)^k B_{n-j} A_{m+j}\Big) = \sum_{l=0}^{k-1} \binom{m+\Delta_A-1}{k-1-l} C_{m+n}^{(l)}. \tag{26}$$

[4] One should use the identity $\binom{0}{n} = \delta_{0,n}$ in the derivation.

Here $k \in \mathbb{Z}$ is equal to the number of operator product expansion singular terms taken into account and can be any integer. The derivation of the commutator formula is based on the same standard trick: double integration (22) of the following expression

$$f(z,w) = A(z)B(w)(z-w)^{\alpha-k}(z/w)^{\gamma}w^{n+\Delta(C^{(0)})+k-2} \tag{27}$$

in two ways as described in the previous section. Then the first way of integration results in the left hand side of (25), the second way gives the right hand side.

At this point it becomes clear why we call the factor μ a commutation factor: it appears when one commutes two field modes.

Infinite sums are involved on the left hand side of commutator formulas (25,26). However as usual the quadratic terms are "nicely" ordered: the modes with "very large" indices are on the right. So if one acts by these infinite sum operators on any state in the highest weight representation, the sums are truncated and become finite.

In the case of usual conformal algebras ($\alpha \in \mathbb{Z}$) taking $k = \alpha$ one recovers the standard commutator formula which can be found elsewhere.

Now we want to discuss how many terms in the operator product expansion one should take and what are the relations between the commutator formulas with different k. The commutation relations (26) with smaller k can always be obtained from those with larger k. So the larger k relations are more informative, which is reasonable, since more terms in the operator product expansion are taken into account. The commutation relations are used to build representation theory of the algebra, e.g. we want to be able to exchange the modes in order to calculate the expectation values of products of the modes. For that it is sufficient to know all the singular terms (i.e. of order $(z-w)^{-n},\ n>0$) in the operator product expansion of any two fields in the theory. So for the needs of representation theory it is sufficient to take k a largest integer number smaller then the singularity α, smaller number of terms may be insufficient to build a meaningful representation theory.

5 Classical $\mathbb{Z}_N$ parafermions

Here we present the classical example of parafermionic conformal field theory, discovered in the pioneering work [1]. These theories are graded by $\mathbb{Z}_N$ group and coincide with the coset

$$\frac{sl(2)_N}{u(1)}, \tag{28}$$

the central charge of which is equal to

$$c = 2\,\frac{N-1}{N+2}. \tag{29}$$

The algebraic data is the following: there are N fields ψ_i of conformal dimensions

$$\Delta_i = \Delta_{N-i} = \frac{i(i-N)}{N}, \qquad i = 0,1,2,\ldots,N-1, \tag{30}$$

the field ψ_0 is just the identity field. The operator product expansions are $\mathbb{Z}_N$ graded:

$$\psi_i(z)\psi_j(w) = \frac{c_{i,j}}{(z-w)^{\alpha_{i,j}}}\Big(\psi_{i+j}(w) + O(z-w)\Big), \tag{31}$$

where all the indices are taken modulo N and the singularity is

$$\alpha_{i,j} = \Delta_i + \Delta_j - \Delta_{i+j}. \tag{32}$$

The commutation factors are easily calculated using the commutator factor relation (10), they are all equal to 1:

$$\mu_{i,j} = 1, \tag{33}$$

and it follows that

$$c_{j,i} = c_{i,j}. \tag{34}$$

One can check that the Jacobi identities (20) are satisfied up to the terms explicitly specified in the operator product expansions (31) if all the structure constants are fixed to the values, calculated in the paper [1] (using the method of correlation functions). The results can be found there. One also obtains from the Jacobi relations the next to leading term in the expansion (31):

$$\big[\psi_i\psi_j\big]_{\alpha_{i,j}-1} = \frac{\Delta_{i+j} + \Delta_i - \Delta_j}{2\Delta_{i+j}}\partial\psi_{i+j}. \tag{35}$$

This term is also explicitly written in the Appendix A of [1]. If $i + j = N$ then this term vanishes. The next term in the operator product expansion of two conjugate fields ψ_i and ψ_{N-i} has conformal dimension 2, and we study from the Jacobi identities that all these terms $\big[\psi_i\psi_{N-i}\big]_{\alpha_{i,N-i}-2}$, $1 \le i \le N/2$ are proportional to the same field. This dimension 2 field can be identified with an energy–momentum field $T(z)$ of the theory. So the ope of conjugate fields takes the form

$$\psi_i(z)\psi_{N-i}(w) = \frac{1}{(z-w)^{2\Delta_i}}\Big(1 + \frac{2\Delta_i}{c}T(w)(z-w)^2 + O((z-w)^3)\Big), \tag{36}$$

where we have fixed the normalization of the parafermionic fields: $c_{i,N-i} = 1,\ 0 < i \le N/2$. The field T is not an independent field since it is equal to the nonsingular term in the operator product expansion of $\psi_1(z)$ and $\psi_{N-1}(w)$. It satisfies the Virasoro algebra of central charge c.

As a simple example of the use of the generalized Jacobi identities (20) we show how to calculate the value of the central charge. We take $A = B = \psi_1, C = \psi_{N-1}$, $\gamma_{AB} = \alpha_{1,1}, \gamma_{BC} = 2\Delta_1 - 3, \gamma_{AC} = 2\Delta_1 - 1$ in (20) and get the formula (29).

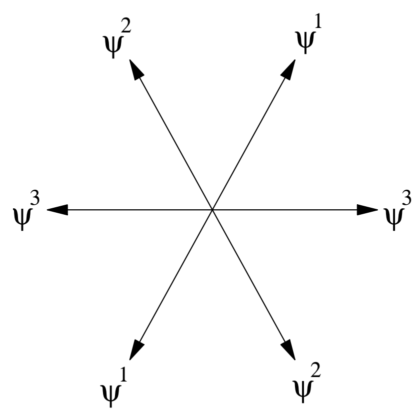
Jacobi identities lead also to additional relations between the next terms in the operator product expansions of basic fields. In particular all the singular terms in the operator product expansions of basic fields are related to the nonsingular terms in the operator product expansions of other basic fields, which makes the representation

theory of the algebra reasonable. There are also relations between nonsingular terms in the operator product expansions.

We would like to note here that other $\mathbb{Z}_N$ parafermionic theories exist, some of them containing commutation factors different from 1 (e.g. the algebras in [6], [7], [14]).

6 $sl(3)$ fermions

Here we discuss a very interesting example of a parafermionic conformal field theory. It is formed by 3 copies of standard free fermions, coupled one to each other by parafermionic type relations. The theory is given by the following coset:



$$\frac{sl(3)_2}{u(1)^2}, \tag{37}$$

and so the central charge of this theory is

$$c = \frac{6}{5}. \tag{38}$$

The theory is only a special case of general coset construction $\mathfrak{g}/u(1)^r$ introduced and studied in [4]. Here $\mathfrak{g}$ is a simple Lie algebra, r is its rank. However we are convinced that the theory deserves a special study.

There are three fields of dimension 1/2: $\psi^{(i)},\ i = 1, 2, 3$, corresponding to the 3 pairs of opposite roots of $sl(3)$. The operator product expansion of each field with itself is the standard free fermion relation:

$$\psi^{(\alpha)}(z)\psi^{(\alpha)}(w) = \frac{1}{z-w} + O(z-w). \tag{39}$$

The operator product expansion of two different fields gives the third one:

$$\psi^{(\alpha)}(z)\psi^{(\beta)}(w) = \frac{c_{\alpha,\beta}\psi^{(\gamma)}(w)}{(z-w)^{1/2}} + O((z-w)^{1/2}), \quad \alpha \neq \beta \neq \gamma. \tag{40}$$

The algebra is obviously $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded. We will assign the following gradings to the fields: $\psi^{(1)}$ grading is $(1, 0)$, $\psi^{(2)}$ grading is $(0, 1)$, $\psi^{(3)}$ grading is $(1, 1)$, and the identity field is $(0, 0)$ graded.

The structure constants are not all independent. The axiom (5) implies

$$c_{\alpha,\beta} = \mu_{\alpha,\beta} c_{\beta,\alpha}. \tag{41}$$

There are 6 parameters $(\mu_{1,2}, \mu_{2,3}, \mu_{3,1}, c_{1,2}, c_{2,3}, c_{3,1})$ to be fixed by the Jacobi identities (20). All the identities are satisfied if and only if

$$\begin{array}{l} \mu_{1,2} = \mu_{2,3} = \mu_{3,1} = x^2, \\ c_{1,2} = c_{2,3} = c_{3,1} = \dfrac{x}{\sqrt{2}}, \end{array} \qquad x = \mathrm{e}^{\pm\frac{\mathrm{i}\pi}{4}}, \mathrm{e}^{\pm\frac{3\mathrm{i}\pi}{4}}. \tag{42}$$

We will make the choice $x = \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$ here, then one obtains the commutation factors: $\mu_{1,2} = \mu_{2,3} = \mu_{3,1} = -\mathrm{i} = -\mu_{2,1} = -\mu_{3,2} = -\mu_{1,3}$.

To calculate the relations between the modes of the fields one has to use the commutator formula (26), the result is:

$$\psi_n^{(\alpha)}\psi_m^{(\alpha)} + \psi_m^{(\alpha)}\psi_n^{(\alpha)} = \delta_{0,m+n}, \tag{43}$$

$$\sum_{j=0}^{\infty} \binom{j-1/2}{j} \left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \psi_{m-1/2-j}^{(\alpha)} \psi_{n+1/2+j}^{(\beta)} + \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} \psi_{n-j}^{(\beta)} \psi_{m+j}^{(\alpha)} \right) = \frac{1}{\sqrt{2}} \psi_{m+n}^{(\gamma)}, \tag{44}$$

where $m, n \in \mathbb{Z}/2$ and in the second relation α, β, γ are all different and ordered: $(\alpha, \beta, \gamma) = (1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$. The exact form of these commutation relations appears here for the first time.

Unitarity can be introduced by the standard conjugation relation:

$$(\psi_n^{(\alpha)})^\dagger = \psi_{-n}^{(\alpha)}, \tag{45}$$

the generalized commutation relations above are invariant under this conjugation.

Next we would like to obtain the expression for the energy-momentum field of the theory. We know the energy-momentum fields of the three free fermion subalgebras: $T^{(\alpha)} = -\frac{1}{2}\left[\psi^{(\alpha)}\partial\psi^{(\alpha)}\right]_0 = \frac{1}{2}\left[\psi^{(\alpha)}\psi^{(\alpha)}\right]_{-1}$. Let us calculate the conformal weight of one of the parafermion fields with respect to another energy-momentum field, i.e. we want to calculate $\left[T^{(\alpha)}\psi^{(\beta)}\right]_2$ for $\alpha \neq \beta$. Substitute in the generalized Jacobi identity (20) $A = B = \psi^{(\alpha)}, C = \psi^{(\beta)}, \gamma_{AB} = -2, \gamma_{BC} = \gamma_{AC} = -1/2$ to get that

$$\left[T^{(\alpha)}\psi^{(\beta)}\right]_2 = \frac{1}{16}\psi^{(\beta)}, \quad \alpha \neq \beta. \tag{46}$$

This result shows that two free fermions lie in the twisted representation of the subalgebra generated by the third one. It is easy to guess that the total energy-momentum tensor T should be proportional to the sum of the three energy-momentum fields. And the factor is easily calculated from the condition $\left[T\psi^{(\alpha)}\right]_2 = \frac{1}{2}\psi^{(\alpha)}$. We conclude that the total energy-momentum field is

$$T = \frac{4}{5}\sum_{\alpha=1}^{3} T^{(\alpha)} = -\frac{2}{5}\sum_{\alpha=1}^{3} {:}\psi^{(\alpha)}\partial\psi^{(\alpha)}{:}, \tag{47}$$

where : : is the standard normal ordering. We leave as an exercise to check that the field T above satisfies the operator product expansion relation of the Virasoro algebra with central charge $c = 6/5$.

The three point correlation function of the parafermionic fields is

$$<\psi^{(\alpha)}(z_1)\psi^{(\beta)}(z_2)\psi^{(\gamma)}(z_3)> = \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}\epsilon_{\alpha\beta\gamma}}}{(z_1 - z_2)^{1/2}(z_2 - z_3)^{1/2}(z_1 - z_3)^{1/2}}, \tag{48}$$

where $\epsilon_{\alpha\beta\gamma}$ is the standard antisymmetric tensor and we still follow the choice $x = \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$.

We would like to say a few words about representation theory of the $sl(3)$ fermion algebra. The main tool in the construction of the representation theory is the commutation relations (43, 44). A highest weight representation is constructed by applying creation operators (nonpositive modes of the three generating fermion fields) to a highest weight state. A highest weight state is a state which is annihilated by all positive modes of the generating fields. The highest weight state of the algebra is also a highest weight state of the free fermion subalgebras. The free fermion algebra has two highest weight representations: the vacuum representation and the twisted representation. So the highest weight representation of the full $sl(3)$ fermion algebra should be a combination of these two representations. It is easy to see from (43, 44) that not all the combinations are allowed. The consistent representations are build from the vacuum highest weight state $|0,0,0>$, which is a vacuum state with respect to all three subalgebras, and from the states $|0,\sigma,\sigma>$, $|\sigma,0,\sigma>$, $|\sigma,\sigma,0>$, which are vacuum states with respect to one of the free fermion subalgebras and twisted states with respect to two other subalgebras. The conformal dimension of the vacuum representation is 0, the conformal dimension of the second type representation is $4/5(1/16+1/16) = 1/10$.

We would like to list here a few examples of states in the vacuum representation:

$$\begin{aligned} &\psi^{(\beta)}_{-5/2}\psi^{(\alpha)}_{-3/2}\psi^{(\alpha)}_{-1/2}|0,0,0>, \\ \psi^{(3)}_{3}\psi^{(1)}_{-4}&\psi^{(3)}_{-5/2}\psi^{(3)}_{-3/2}\psi^{(2)}_{1}\psi^{(1)}_{-3/2}|0,0,0>, \\ \psi^{(2)}_{-1/2}&\psi^{(1)}_{5/2}\psi^{(3)}_{-5}\psi^{(1)}_{-3}\psi^{(1)}_{-2}\psi^{(2)}_{-7/2}|0,0,0>. \end{aligned} \tag{49}$$

The rule for the choice of integer or half-integer modes is the following: as one adds operators from the left at any stage the sum of modes should be integer, if the $\mathbb{Z}_2\times\mathbb{Z}_2$ charge of the string of operators is $(0,0)$, and the sum of modes should be half-integer otherwise.

The examples of states in the representation generated from the highest weight state $|0,\sigma,\sigma>$ are

$$\begin{aligned} &\psi^{(3)}_{3/2}\psi^{(1)}_{-4}\psi^{(3)}_{-5}\psi^{(3)}_{-3}\psi^{(2)}_{1/2}\psi^{(1)}_{-3/2}|0,\sigma,\sigma>, \\ &\psi^{(2)}_{-2}\psi^{(1)}_{5/2}\psi^{(3)}_{-5/2}\psi^{(1)}_{-3}\psi^{(1)}_{-2}\psi^{(2)}_{-7}|0,\sigma,\sigma>. \end{aligned} \tag{50}$$

The rule is basically the same as above, the only difference is that one should also count the state $|0,\sigma,\sigma>$ itself as it would carry a half-integer mode and a $\mathbb{Z}_2\times\mathbb{Z}_2$ grading $(1,0)$.

As a last remark on the representation theory we would like to note that a kind of Poincare–Birkhoff–Witt (PBW) theorem should hold, but we are even not sure how to choose the PBW basis in the case of representations discussed above.

A few words about parafermionic models of higher level cosets:

$$\frac{sl(3)_N}{u(1)^2}. \tag{51}$$

The parafermionic algebra becomes $\mathbb{Z}_N \times \mathbb{Z}_N$ graded. It is generated by N^2 fields $\psi^{(i,j)}$, $i, j = 0, 1, \ldots, N-1$ with $\psi^{(0,0)}$ being the identity field. The conformal dimensions are given by the following formula

$$\Delta_{i,j} = \max(i,j) - \frac{i^2 + j^2 - i\,j}{N}\,. \tag{52}$$

The operator product expansion of two fields $\psi^{(i,j)}$ and $\psi^{(k,l)}$ gives the field $\psi^{(i+k,j+l)}$, where all the indices are taken modulo N. A study of generalized Jacobi identities reveals that all the commutation factors are $2N$-roots of unity, i.e. $\mu^{2N} = 1$, and that some structure constants vanish if N is odd. These $\mathbb{Z}_N \times \mathbb{Z}_N$ graded algebras contain many $\mathbb{Z}_N$ graded subalgebras, different from those which have ever been studied.

7 Simply laced fermions

In this section we generalize the $sl(3)$ fermion algebra from the previous section. The underlying root system is the root system of a simple Lie algebra $\mathfrak{g}$ of A-D-E type. The corresponding coset is

$$\frac{\mathfrak{g}_2}{u(1)^r}, \tag{53}$$

where r is the rank of the algebra $\mathfrak{g}$.

The roots of a simply laced Lie algebra are all of the same length. Two root directions may be either orthogonal or having the angle $\pi/3$ between them. We associate to every root direction α a free fermion algebra generated by a field $\psi^{(\alpha)}$:

$$\psi^{(\alpha)}(z)\psi^{(\alpha)}(w) = \frac{1}{z-w} + O(z-w). \tag{54}$$

Two fields are not coupled if the corresponding root directions are orthogonal. If the root directions are not orthogonal then the fields are coupled in a parafermionic way:

$$\psi^{(\alpha)}(z)\psi^{(\beta)}(w) = \begin{cases} O((z-w)^0), & \alpha \text{ and } \beta \text{ are orthogonal,} \\ \frac{c_{\alpha,\beta}\psi^{(\alpha+\beta)}(w)}{(z-w)^{1/2}} + O((z-w)^{1/2}), & \alpha \text{ and } \beta \text{ are not orthogonal.} \end{cases} \tag{55}$$

It is not clear what is the grading abelian group in the case of algebraic relations described above. Since due to (54) the square of each element is identity, the grading group should be $\mathbb{Z}_2^k$ for some integer k. However the number of generating fields in the theory is in general less than 2^k.

The structure constants $c_{\alpha,\beta}$ and the commutation factors $\mu_{\alpha,\beta}$ are to be fixed by the Jacobi identities. In fact there is a number of solutions to the Jacobi identities, although we expect that all of them are equivalent from the physical point of view. If the root directions α and β are not orthogonal, then $\alpha, \beta, \alpha + \beta$ form an $sl(3)$ root subsystem, and therefore the structure constants and the commutation factors for the

corresponding $sl(3)$ fermion subalgebra are subject to the relations (42). The commutation factors between the orthogonal fermions are equal ± 1. The naive expectation that the fermions which are not coupled are all mutually anticommutative is not true in general: such solution exists only in the case $\mathfrak{g} = sl(4)$ and for the higher rank algebras the commutation factors for the orthogonal root directions can not be chosen all of the same sign!

We will present explicitly one of the solutions for the case $\mathfrak{g} = sl(n)$, for general n. The root directions of $sl(n)$ satisfy the "$so(n)$ pattern" in the sense that the fusion rules are identical to those of the $so(n)$ algebra in the standard basis of antisymmetric $n \times n$ matrices. So it would be convenient to label the fields by a pair of two numbers: $\psi^{(i,j)}$, $1 < i \neq j < n$. The order of indices is not important ($\psi^{(i,j)} \equiv \psi^{(j,i)}$), so we will always assume that $i < j$. In this two index notation the $sl(3)$ fermions from the previous section are written as $\psi^{(1)} = \psi^{(2,3)}, \psi^{(2)} = \psi^{(1,3)}, \psi^{(3)} = \psi^{(1,2)}$.

If two fields have no common indices then they are not coupled. If one of the indices is common for two root directions then these root directions are not orthogonal and their sum is labelled by the two distinct indices from the two pairs, for example $(1,2) + (1,4) = (2,4)$ and the corresponding operator product expansion is

$$\psi^{(1,2)}(z)\psi^{(1,4)}(w) = \frac{c_{(1,2),(1,4)}\ \psi^{(2,4)}}{(z-w)^{1/2}} + O((z-w)^{1/2}). \tag{56}$$

We express the structure constants and the commutation factors again in terms of one variable x, which can take one of the four values $x = \mathrm{e}^{\pm \frac{\mathrm{i}\pi}{4}}, \mathrm{e}^{\pm \frac{3\mathrm{i}\pi}{4}}$. In all the formulas below we assume that $(i,j) < (k,l)$, that means $i < k$ or $(i = k, j < l)$. The structure constants are

$$c_{(i,j),(k,l)} = \begin{cases} x/\sqrt{2}, & i = k \text{ or } j = l, \\ 1/(x\sqrt{2}), & i = l \text{ or } j = k. \end{cases} \tag{57}$$

If one index is common then the commutation factor is

$$\mu_{(i,j),(k,l)} = \begin{cases} x^2, & i = k \text{ or } j = l, \\ 1/(x^2), & i = l \text{ or } j = k. \end{cases} \tag{58}$$

If all the indices are different then

$$\mu_{(i,j),(k,l)} = \epsilon_{ijkl}\ , \tag{59}$$

where ϵ_{ijkl} is antisymmetric in all 4 indices and it is equal to 1 when $i < j < k < l$. So we see, that the commutation factor of two coupled fields is $\pm\mathrm{i}$, and the commutation factor of two fields which are not coupled is ± 1.

To satisfy the Jacobi identities one should also require the following condition

$$\left[\psi^{(i,j)}\psi^{(k,l)}\right]_0 + \mathrm{i}\left[\psi^{(i,k)}\psi^{(j,l)}\right]_0 - \left[\psi^{(i,l)}\psi^{(j,k)}\right]_0 = 0, \quad i < j < k < l. \tag{60}$$

The commutation relations read

$$\psi_n^{(i,j)}\psi_m^{(i,j)} + \psi_m^{(i,j)}\psi_n^{(i,j)} = \delta_{0,m+n}, \tag{61}$$

$$\psi_n^{(i,j)}\psi_m^{(k,l)} - \epsilon_{ijkl}\psi_m^{(k,l)}\psi_n^{(i,j)} = 0, \quad i,j,k,l \text{ are all different}, \tag{62}$$

if the pairs of indices are the same or have no common indices. If two pairs of indices have one common index then the commutation relation between the corresponding field modes is of parafermionic type:

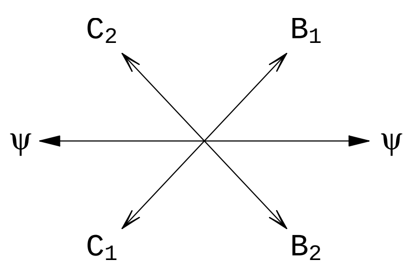
$$\begin{gathered}\sum_{s=0}^{\infty} \binom{s-1/2}{s} \Big(x^{-\epsilon}\psi_{m-1/2-s}^{(i,j)}\psi_{n+1/2+s}^{(k,l)} + x^{\epsilon}\psi_{n-s}^{(k,l)}\psi_{m+s}^{(i,j)} \Big) = \frac{1}{\sqrt{2}}\psi_{m+n}^{(\bar{i},\bar{k})}, \\ (i,j) < (k,l), \quad \epsilon = \begin{cases} 1, & i=k \text{ or } j=l, \\ -1, & i=l \text{ or } j=k, \end{cases}\end{gathered} \tag{63}$$

where $\bar{i}$ and $\bar{k}$ are the two distinct indices from the set $\{i,j,k,l\}$. The condition (60) is translated to the following relation:

$$\sum_{n_1} \psi_{m-n_1}^{(i,j)}\psi_{n_1}^{(k,l)} + \mathrm{i}\sum_{n_2} \psi_{m-n_2}^{(j,k)}\psi_{n_2}^{(j,l)} - \sum_{n_3} \psi_{m-n_3}^{(i,l)}\psi_{n_3}^{(j,k)} = 0. \tag{64}$$

8 $sl(2|1)$ level 2 coset

In this section we introduce a new parafermionic conformal field theory model. It is generated by one dimension 1/2 free fermion field and 4 dimension 1 fields. In the end of the section we will show that this model coincides with the coset of the $sl(2|1)$ affine superalgebra on level 2 by the two $u(1)$ currents, corresponding to the Cartan elements of the $sl(2|1)$ algebra.



The fields are associated to the roots of the $sl(2|1)$ root system as shown on the root diagram. The field ψ is again the standard free fermion field:

$$\psi(z)\psi(w) = \frac{1}{z-w} + O(z-w). \tag{65}$$

Each pair of the dimension 1 fields, (B_1, C_1) and (B_2, C_2) forms the $psl(1|1)$ current algebra:

$$B_i(z)C_i(w) = \frac{1}{(z-w)^2} + O\big((z-w)^0\big), \tag{66}$$

$$B_i(z)B_i(w) = O\big(z-w\big), \tag{67}$$

$$C_i(z)C_i(w) = O\big(z-w\big), \tag{68}$$

where $i = 1$ or 2. The mutual commutation factor is

$$\mu_{B_i,C_i} = -1, \tag{69}$$

which means that the fields are anticommutative and therefore

$$C_i(z)B_i(w) = -\frac{1}{(z-w)^2} + O\big((z-w)^0\big). \tag{70}$$

The root diagram tells us what should be the fusion rules between the fields. The operator product expansion of two fields associated with the roots α and β gives the field associated with the root $\alpha+\beta$ if there is such a root, and the operator product expansion is not singular if $\alpha+\beta$ is not a root. So the relations are

$$B_1(z)B_2(w) = \kappa_{B_1,B_2}\frac{\psi(w)}{(z-w)^{3/2}} + O\big((z-w)^{-1/2}\big), \tag{71}$$

$$C_1(z)C_2(w) = \kappa_{C_1,C_2}\frac{\psi(w)}{(z-w)^{3/2}} + O\big((z-w)^{-1/2}\big), \tag{72}$$

$$B_1(z)C_2(w) = O\big((z-w)^{1/2}\big), \tag{73}$$

$$C_1(z)B_2(w) = O\big((z-w)^{1/2}\big), \tag{74}$$

$$\psi(z)B_1(w) = \kappa_{\psi,B_1}\frac{C_2(w)}{(z-w)^{1/2}} + O\big((z-w)^{1/2}\big), \tag{75}$$

$$\psi(z)C_2(w) = \kappa_{\psi,C_2}\frac{B_1(w)}{(z-w)^{1/2}} + O\big((z-w)^{1/2}\big), \tag{76}$$

$$\psi(z)B_2(w) = \kappa_{\psi,B_2}\frac{C_1(w)}{(z-w)^{1/2}} + O\big((z-w)^{1/2}\big), \tag{77}$$

$$\psi(z)C_1(w) = \kappa_{\psi,C_1}\frac{B_2(w)}{(z-w)^{1/2}} + O\big((z-w)^{1/2}\big), \tag{78}$$

where the coefficients κ are structure constants.

We have checked the Jacobi identities to obtain the commutation factors:

$$\mu_{B_1,B_2} = \mu_{C_1,C_2} = \mu_{B_1,C_2} = \mu_{C_1,B_2} = \mu_{\psi,B_2} = \mu_{\psi,C_2} = -\mu_{\psi,B_1} = -\mu_{\psi,C_1} = -\mathrm{i}, \tag{79}$$

and the structure constants:

$$\kappa_{\psi,B_1} = \kappa_{\psi,C_1} = -\kappa_{B_1,B_2} = \kappa_{C_1,C_2} = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2}}, \quad \kappa_{\psi,B_2} = \kappa_{\psi,C_2} = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}}. \tag{80}$$

In the relations above the following singular terms are not specified: $\big[B_1B_2\big]_{1/2}$ and $\big[C_1C_2\big]_{1/2}$. One has to take these terms into account in order to be able to write the meaningful generalized commutation relations between B_1 and B_2 and between C_1 and C_2. The Jacobi identities imply that these terms are proportional, so only one new field

has to be introduced. It has conformal dimension 3/2, we will call it G. The full form of operator product expansions (71) and (72) is

$$B_1(z)B_2(w) = \kappa_{B_1,B_2}\left(\frac{\psi(w)}{(z-w)^{3/2}} + \frac{\frac{1}{2}\partial\psi(w) - \mathrm{i}\frac{\sqrt{3}}{2}G(w)}{(z-w)^{1/2}}\right) + O\big((z-w)^{1/2}\big), \qquad (81)$$

$$C_1(z)C_2(w) = \kappa_{C_1,C_2}\left(\frac{\psi(w)}{(z-w)^{3/2}} + \frac{\frac{1}{2}\partial\psi(w) - \mathrm{i}\frac{\sqrt{3}}{2}G(w)}{(z-w)^{1/2}}\right) + O\big((z-w)^{1/2}\big). \qquad (82)$$

The operator product expansions of the field G with other basic fields read

$$\psi(z)G(w) = O\big((z-w)^0\big), \qquad (83)$$

$$\begin{aligned} G(z)G(w) = {} & \frac{-\frac{5}{3}}{(z-w)^3} + \\ & + \frac{-\frac{4}{3}\big[B_1C_1\big]_0(w) - \frac{4}{3}\big[B_2C_2\big]_0(w) + \frac{1}{3}\big[\psi\psi\big]_{-1}(w)}{z-w} + O\big((z-w)^0\big), \end{aligned} \qquad (84)$$

$$\begin{aligned} G(z)B_{1,2}(w) = {} & \frac{\frac{5}{2\sqrt{6}}\mathrm{e}^{\pm\mathrm{i}\pi/4}C_{2,1}(w)}{(z-w)^{3/2}} + \\ & + \frac{\sqrt{\frac{2}{3}}\mathrm{e}^{\pm\mathrm{i}\pi/4}\partial C_{2,1}(w) \mp \frac{\mathrm{i}}{2\sqrt{3}}\big[\psi B_{1,2}\big]_{-1/2}(w)}{(z-w)^{1/2}} + O\big((z-w)^{1/2}\big), \end{aligned} \qquad (85)$$

$$\begin{aligned} G(z)C_{1,2}(w) = {} & \frac{\frac{5}{2\sqrt{6}}\mathrm{e}^{\pm\mathrm{i}\pi/4}B_{2,1}(w)}{(z-w)^{3/2}} + \\ & + \frac{\sqrt{\frac{2}{3}}\mathrm{e}^{\pm\mathrm{i}\pi/4}\partial B_{2,1}(w) \mp \frac{\mathrm{i}}{2\sqrt{3}}\big[\psi C_{1,2}\big]_{-1/2}(w)}{(z-w)^{1/2}} + O\big((z-w)^{1/2}\big), \end{aligned} \qquad (86)$$

The commutation factors of G are related to the commutation factors of ψ:

$$\mu_{G,A} = -\mu_{\psi,A}, \qquad (87)$$

where A is any of the basic fields $\psi, G, B_1, C_1, B_2, C_2$. In particular $\mu_{\psi,G} = -1$, i.e. the fields ψ and G are anticommutative.

The Jacobi identities are satisfied modulo a null field condition

$$\big[B_1C_1\big]_0 - \big[B_2C_2\big]_0 - \mathrm{i}\frac{\sqrt{3}}{2}\big[\psi G\big]_0 = 0. \qquad (88)$$

The algebra of 6 basic fields $\psi, B_1, C_1, B_2, C_2, G$ is closed in the sense that all the singular terms in the operator product expansions of two basic fields are expressed in terms of basic fields, their derivatives and composite fields. By “composite field” we understand a field which is equal to a nonsingular term in the operator product expansion of two basic fields.

The algebra has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading and a $U(1)$ charge. The fields B_1, C_1 have grading $(0,1)$, B_2, C_2 - $(1,0)$, ψ, G - $(1,1)$. The fields B_1 and C_2 have charge $+1$, B_2 and C_1 have charge -1, and ψ, G - charge 0.

The energy-momentum field of the theory is

$$T = \frac{2}{3}\left(\left[\psi\psi\right]_{-1} - \left[B_1 C_1\right]_0 - \left[B_2 C_2\right]_0\right). \tag{89}$$

It satisfies the Virasoro algebra of central charge

$$c = -2. \tag{90}$$

The field $G(z)$ generates the $N = 1$ superconformal algebra of central charge $-5/2$, the corresponding Virasoro field is $T_G = \left[GG\right]_1/2$. The full Virasoro algebra decouples to a sum of two commuting parts: $T = T_G + T_\psi$, there $T_\psi = \left[\psi\psi\right]_{-1}/2$ is the Virasoro field of the free fermion subalgebra.

The lengthy generalized commutation relations are

$$\psi_n \psi_m + \psi_m \psi_n = \delta_{n+m,0}, \tag{91}$$

$$(B_i)_n (C_i)_m + (C_i)_m (B_i)_n = n\delta_{n+m,0}, \tag{92}$$

$$\sum_{j=0}^{\infty} \binom{j-1/2}{j} \left(\mathrm{e}^{\pm\frac{\mathrm{i}\pi}{4}} \psi_{m-1/2-j} (B_{1,2})_{n+1/2+j} - \mathrm{e}^{\mp\frac{\mathrm{i}\pi}{4}} (B_{1,2})_{n-j} \psi_{m+j}\right) = \frac{1}{\sqrt{2}} (C_{2,1})_{m+n}, \tag{93}$$

$$\sum_{j=0}^{\infty} \binom{j-1/2}{j} \left(\mathrm{e}^{\pm\frac{\mathrm{i}\pi}{4}} \psi_{m-1/2-j} (C_{1,2})_{n+1/2+j} - \mathrm{e}^{\mp\frac{\mathrm{i}\pi}{4}} (C_{1,2})_{n-j} \psi_{m+j}\right) = \frac{1}{\sqrt{2}} (B_{2,1})_{m+n}, \tag{94}$$

$$\sum_{j=0}^{\infty} \binom{j-1/2}{j} \left((B_1)_{m-1/2-j} (C_2)_{n+1/2+j} + \mathrm{i}(C_2)_{n-j} (B_1)_{m+j}\right) = 0, \tag{95}$$

$$\sum_{j=0}^{\infty} \binom{j-1/2}{j} \left((C_1)_{m-1/2-j} (B_2)_{n+1/2+j} + \mathrm{i}(B_2)_{n-j} (C_1)_{m+j}\right) = 0, \tag{96}$$

$$\begin{aligned}
&\sum_{j=0}^{\infty} \binom{j-1/2}{j} \left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} (C_1)_{m-1/2-j} (C_2)_{n+1/2+j} - \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} (C_2)_{n-j} (C_1)_{m+j}\right) = \\
&= -\sum_{j=0}^{\infty} \binom{j-1/2}{j} \left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} (B_1)_{m-1/2-j} (B_2)_{n+1/2+j} - \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} (B_2)_{n-j} (B_1)_{m+j}\right) = \\
&= \frac{1}{2\sqrt{2}} \left((m - n - \frac{1}{2})\psi_{m+n} - \mathrm{i}\sqrt{3} G_{m+n}\right),
\end{aligned} \tag{97}$$

$$\psi_n G_m + G_m \psi_n = 0, \tag{98}$$

where we omit all the other generalized commutation relations involving field G, since the expressions are too long. The mode expansions of the composite fields should be calculated using the same generalized commutator formula (26).

Now we want to show that the above algebra describes the $sl(2|1)_2/u(1)^2$ coset. We introduce two commuting free bosons ϕ_1 and ϕ_2:

$$\begin{aligned} \phi_1(z)\phi_1(w) &= -\log(z-w), \\ \phi_2(z)\phi_2(w) &= \log(z-w). \end{aligned} \tag{99}$$

(Note the sign difference!) Then the currents of the $sl(2|1)$ affine algebra are expressed as

$$\begin{aligned} &H_1(z) \sim \mathrm{i}\partial\phi_1(z), && J^+(z) \sim \psi(z)\mathrm{e}^{\mathrm{i}\phi_1(z)}, \\ &H_2(z) \sim \mathrm{i}\partial\phi_2(z), && J^-(z) \sim \psi(z)\mathrm{e}^{-\mathrm{i}\phi_1(z)}, \\ &F^+_{1,2}(z) \sim B_{1,2}(z)\mathrm{e}^{\frac{\mathrm{i}}{2}(\phi_1(z)\pm\phi_2(z))}, && F^-_{1,2}(z) \sim C_{1,2}(z)\mathrm{e}^{\frac{\mathrm{i}}{2}(-\phi_1(z)\pm\phi_2(z))}. \end{aligned} \tag{100}$$

There are 4 bosonic currents (H_1, H_2, J^+, J^-) and 4 fermionic currents ($F_1^+, F_1^-, F_2^+, F_2^-$), they form an $sl(2|1)$ affine algebra on level 2. H_1, H_2 correspond to the two Cartan elements of $sl(2|1)$, J^+, J^- - to the two even roots, $F_1^+, F_1^-, F_2^+, F_2^-$ - to the 4 odd roots. The fields H_1, J^+, J^- generate the $sl(2)$ affine subalgebra on level 2.

9 Summary

We described conformal field theories of parafermionic type from the algebraic point of view. The main algebraic tool, the Jacobi type identity, is presented in the explicit form. Using this identity we calculated the commutation factors and the structure constants for the following cosets: $sl(3)_2/u(1)^2$, $sl(N)_2/u(1)^2$, $sl(2|1)_2/u(1)^2$. We wrote the generalized commutation relations for these parafermionic algebras, and studied the representation theory of the algebra of the $sl(3)_2/u(1)^2$ coset theory. The theories corresponding to the $sl(3)_2/u(1)^2$, $sl(N)_2/u(1)^2$ cosets were known since a work by Gepner [4], but the exact algebraic relations between the fields were never studied in the past.

The results above demonstrate the power of algebraic approach in the study of parafermionic conformal field theories. The methods described will hopefully lead to discovery of new algebras, and new two-dimensional conformal models.

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